EFFECT OF VARIABILITY OF THE MODULUS OF ELASTICITY ON THE THERMOELASTIC STRESSES ASSOCIATED WITH THE PULSED HEATING OF A ROD

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In the practical utilization of thermal shock in a rod with a load at the end in accordance with the method of [1] as a means of studying the stability of the rod and the dynamic modulus of elasticity of elevated temperatures, it is necessary under certain conditions to take into account the effect of the variability of the modulus of elasticity on the thermoelastic forces. This is the subject of the present article.

Consider the behavior of an elastic rod of length l and mass m fixed at one end x = 0 and with a concentrated mass M at the other end x = l in the presence of rapid heating. Since the modulus of elasticity of the rod material depends on the temperature, which, in turn, is a function of time, in what follows we assume that the modulus of elasticity E = E(t) is a function of time.

The dynamic equilibrium equation for the mass M has the form:

$$Mu^{\cdot\cdot} + \sigma F = 0 \tag{1}$$

Here, u is the displacement of the end of the rod with the load M in the axial direction, σ is the longitudinal stresses at that end, F is the cross-sectional area of the rod, and a dot denotes differentiation with respect to time t.

Neglecting thermoelastic interaction and using the stress-strain relation

$$\sigma(t) = E(t)\,[l^{-1}u(t) - \alpha\, T(t)] \tag{2}$$

where α is the coefficient of linear expansion and T(t) the temperature rise at time t, we reduce the equilibrium equation to the form

$$Mu^{\cdot\cdot} + E(t)Fl^{-1}u = E(t)F\alpha\, T(t) \tag{3}$$

Assume that the rod is uniformly heated over the length and cross section and that the temperature rise can be expressed by the equation

$$T(t) = T_+ t / t^* \;(0 \leqslant t \leqslant t^*), \qquad T(t) = T_+ \quad (t > t^*) \tag{4}$$

Here, T_+ is the maximum temperature rise and t* is the heating time. Since for many materials the temperature dependence of the modulus of elasticity can be approximated over a wide interval by a linear function [2], using condition (4) we represent this dependence in the form

$$E(t) = E_0(1 - \lambda)t / t^* \quad (0 \leqslant t \leqslant t^*), \qquad (\lambda = 1 - E_1 / E_0)$$
$$E(t) = E_1 \qquad (t > t^*) \tag{5}$$

Here, $\lambda = 1 - E_1/E_0$ is a parameter and E_0 and E_1 are the moduli of elasticity at the starting temperature and after it has risen by T_+.

Substituting (4) and (5) into Eq. (3), we obtain

$$u^{\cdot\cdot} + \omega^2\eta u = \omega^2\eta l\alpha T_+ t / t^* \tag{6}$$
$$\omega_* = \frac{c}{l}\left(\frac{1}{\gamma}\right)^{1/2}, \quad c = \left(\frac{E_0}{\rho}\right)^{1/2}, \quad \gamma = \frac{m}{M}, \quad \eta = 1 - \lambda t / t^*$$

where ρ is the density of the rod material.

The solution of homogeneous equation (6) is given by the functions [3]

$$u_1(t)=\eta^{1/2}J_{1/3}\left[\frac{2}{3}\frac{\omega t^*}{\lambda}\eta^{3/2}\right],\quad u_2(t)=\eta^{1/2}Y_{1/3}\left[\frac{2}{3}\frac{\omega t^*}{\lambda}\eta^{3/2}\right] \tag{7}$$

Here, $J_{1/3}$ (z) and $Y_{1/3}$ (z) are Bessel functions of the first and second kinds.

For zero initial data the displacement and velocity $u^{\cdot}$ of the end of the rod $x = l$ are given by the expressions:

for $0 \le t \le t^*$

$$\frac{u(t)}{l\alpha T_+}=\frac{\pi}{3\lambda}\eta^{1/2}[J_{1/3}(\beta)Y_{1/3}(\beta\eta^{3/2})-Y_{1/3}(\beta)J_{1/3}(\beta\eta^{3/2})]+\frac{t}{t^*}$$
$$\frac{u^{\cdot}(t)}{l\alpha T_+\omega}=\frac{\pi}{3\lambda}\left[\frac{1}{3\beta}\eta^{-1/2}(Y_{1/3}(\beta)J_{1/3}(\beta\eta^{3/2}))-J_{1/3}(\beta)Y_{1/3}(\beta\eta)^{3/2})\right.$$
$$\left.+\eta(Y_{1/3}(\beta)J^{\cdot}(\beta\eta^{3/2})-J_{1/3}(\beta)Y_{1/3}^{\cdot}(\beta\eta^{3/2}))+\frac{1}{\omega t^*}\right] \tag{8}$$

for $t > t^*$

$$\frac{u(t)}{l\alpha T_+}=1+\frac{\pi}{3}\frac{\varphi^{1/2}}{\lambda}A\left[1+\frac{1}{\varphi^2}\left(\frac{1}{{}^1\!/_2\pi\beta A}-\frac{\varphi^{-1/2}}{{}^1\!/_2\pi\beta A}\frac{J_{1/3}(\beta)}{J_{1/3}(\beta\varphi^{3/2})}\right.\right.$$
$$\left.\left.-\varphi\frac{J_{-2/3}(\beta\varphi^{3/2})}{J_{1/3}(\beta\varphi)^{3/2}}\right)^2\right]^{1/2}\sin[\omega_1(t-t^*)+\delta]$$
$$\frac{u^{\cdot}(t)}{l\alpha T_+\omega_1}=\frac{\pi}{3}\frac{\varphi^{1/2}}{\lambda}A\left[1+\frac{1}{\varphi^2}\left(\frac{1}{{}^1\!/_2\pi\beta A}-\frac{\varphi^{-1/2}}{{}^1\!/_2\pi\beta A}\frac{J_{1/3}(\beta)}{J_{1/3}(\beta\varphi^{3/2})}\right.\right.$$
$$\left.\left.-\varphi\frac{J_{-2/3}(\beta\varphi^{3/2})}{J_{1/3}(\beta\varphi^{3/2})}\right)^2\right]^{1/2}\cos[\omega_1(t-t^*)+\delta] \tag{9}$$

Here,

$$A=[J_{1/3}(\beta)Y_{1/3}(\beta\varphi^{3/2})-J_{1/3}(\beta\varphi^{3/2})Y_{1/3}(\beta)]\quad(\varphi=1-\lambda)$$
$$\beta=\frac{2}{3}\frac{\omega t^*}{\lambda},\quad \omega_1=\frac{c_1}{l}\cdot\left(\frac{1}{\gamma}\right)^{1/2}\quad c_1=\left(\frac{E_1}{\rho}\right)^{1/2},\quad \operatorname{tg}\delta=\frac{u(t^*)-l\alpha T_+}{u^{\cdot}(t^*)/\omega_1}$$

In Eqs. (9) the values of u (t^*) and $u^{\cdot}$ (t^*) are calculated in accordance with (8) for $t = t^*$.

We find the thermoelastic stresses in the rod by means of Eq. (2):

for $0 \le t \le t^*$

$$\frac{\sigma(t)}{E_0\alpha T_+}=\frac{\pi}{3\lambda}\eta^{3/2}[J_{1/3}(\beta)Y_{1/3}(\beta\eta^{3/2})-J_{1/3}(\beta\eta^{3/2})Y_{1/3}(\beta)] \tag{10}$$

for $t > t^*$

$$\frac{\sigma(t)}{E_0\alpha T_+}=\frac{\pi}{3\lambda}\varphi^{3/2}A\left[1+\frac{1}{\varphi^2}\left(\frac{1}{{}^1\!/_2\pi\beta A}-\frac{\varphi^{-1/2}}{{}^1\!/_2\pi\beta A}\frac{J_{1/3}(\beta)}{J_{1/3}(\beta\varphi^{3/2})}\right.\right.$$
$$\left.\left.-\varphi\frac{J_{-2/3}(\beta\varphi^{3/2})}{J_{1/3}(\beta\varphi^{3/2})}\right)^2\right]^{1/2}\sin[\omega_1(t-t^*)+\delta] \tag{11}$$

Thus, the displacement of the end of the rod $x = l$, its velocity, and the stresses in the rod for a variable modulus of elasticity are completely determined by (8)–(11).

Consider the limiting case in which the time t^* is much less than the fundamental period of the natural vibrations of the loaded rod $2\pi/\omega$. In this case in Eq. (9) $\beta \to 0$. Using representations of the Bessel functions in series form [4] and confining ourselves to the first term of the expansion for small values of the argument, we have

$$\frac{u(t)}{l\alpha T_+}=1-\cos\omega_1 t,\qquad \frac{\sigma(t)}{E_1\alpha T_+}=\cos\omega_1 t \tag{12}$$

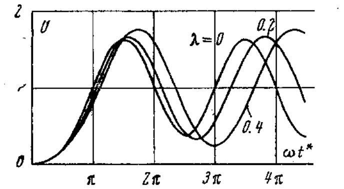

Fig. 1

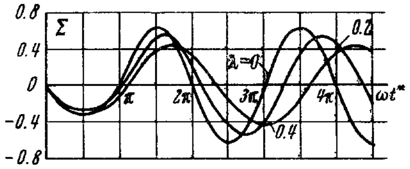

Fig. 2

It follows from expression (12) that in this case, as was to be expected, the natural frequency and the maximum amplitude of the thermoelastic stresses are determined by the value of the modulus of elasticity after heating by T_+. In the figures we have used the notation $U = u/l\alpha T_+$ and $\Sigma = \sigma/E_0\alpha T_+$.

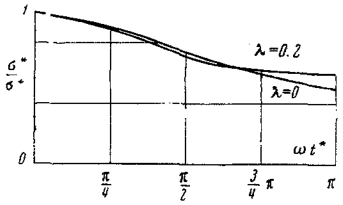

Fig. 3

The effect of the parameter λ and the heating time ωt* on the displacement of the end of the rod and the stresses in it when the vibrations are forced (during heating) and when they are free (after heating) can be seen from Figs. 1 and 2, where the u(t) and σ(t) relations are presented for $\omega t^* = \pi$ and various values of the parameter λ. As λ increases, so does the period of the free vibrations; this is attributable to the decrease in the modulus of elasticity E during heating (see Eq. (1) of [1]). The amplitude of the compressive stresses is less during than after heating (Fig. 2). The ratio of the maximum compressive stress in the first quarter period to the maximum compressive stress after heating σ^*/σ^+ is given in Fig. 3 for various heating times ωt* and parameters λ.

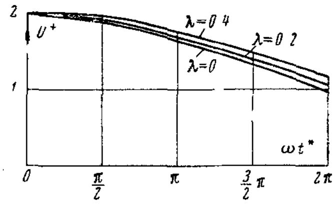

Fig. 4

The dependence of the maximum displacement u^+ and stress σ^+ on the heating time ωt* and the parameter λ is shown in Figs. 4 and 5. From these figures it is clear that at $\omega t^* < {}^1/_2\pi$ the maximum displacements depend only slightly on the parameter λ; the effect of λ on the stresses is approximately proportional to the magnitude of λ.

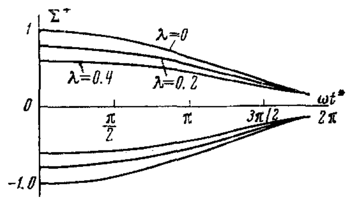

Fig. 5

An analysis of the data presented in Figs. 1 and 2 shows that at a heating time not exceeding a quarter period the displacements and thermoelastic stresses can be calculated without serious error, as for the case of instantaneous heating, from the period of the free vibrations and modulus of elasticity taken at the maximum heating temperature, i.e., from relation (12).

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